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Bi-Hamiltonian structure of N-component Kodama equations

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Abstract. We present a simple way of constructing the second Hamiltonian operators for N-component Kodama equations. Using dimensional analysis we are led to an ansatz for the Hamiltonian operator as well as the conserved quantities in terms of ratios of polynomials. The coefficients of these polynomials are determined from the Jacobi identities. The resulting bi-Hamiltonian structure consists of generalization of Cavalcante and McKean's work for $N = 2$ and our earlier results for $N = 3, 4$.

1. Introduction

The Hamiltonian structure [1] of equations of hydrodynamic type and their differential geometry were introduced by Dubrovin and Novikov in [2]. This theory was originally developed for systems arising from averaging the completely integrable nonlinear evolution equations such as the KdV and Sine–Gordon equations, by the Bogolyubov–Whitham averaging method. We refer to the recent expository article [3] and the extensive bibliography therein.

On the other hand, two-component equations of hydrodynamic type consist of second-order quasilinear partial differential equations which have been studied in the last three centuries. The foremost example of these equations consists of the Eulerian equations of gas dynamics which Nutku [4] has shown to admit tri-Hamiltonian structure. In subsequent papers [5,6] this structure was further investigated. Almost all two-component equations of hydrodynamic type have been realized in the generalized gas dynamics hierarchy with at least quadri-Hamiltonian structure, three of which are of hydrodynamic type and one is of third order.

Yet another way of obtaining hydrodynamic-type equations has been introduced by Kodama [7] as a result of a reduction procedure applied to the dispersionless–Kadomtsev–Petviashvili equation

$$(\phi_t - \phi\phi_x)_x = \phi_{yy} \tag{1}$$

which is also known as the Zabolostkaya–Khokhlov equation. The Kodama reduction gives equations of hydrodynamic type in some auxiliary functions $\{u^i\}$, $i = 1, 2, \dots N$. In $1 + 1$ dimensions these equations are defined by an $N \times N$ matrix $v(u)$

$$u^i_t = v^i_j(u)\, u^j_x \tag{2}$$

where we employ the summation convention over repeated indices, and the set of equations in the other *time* coordinate y is related to (2) in order to ensure the compatibility conditions. For each choice of matrix $v(u)$, these equations of hydrodynamic type are used to construct a class of solutions of the Zabolostkaya–Khokhlov *equation* [7].

Kodama has shown that the shallow-water equations are an example of such a system with

$$v(u) = \begin{pmatrix} u^2 & u^1 \\ 1 & u^2 \end{pmatrix} \tag{3}$$

and he proposed the following generalization:

$$v(u) = \begin{pmatrix} u^N & u^{N-1} & u^{N-3} & \dots & u^1 \\ 0 & u^N & u^{N-1} & \dots & u^2 \\ 0 & 0 & u^N & \dots & u^3 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & u^{N-1} \\ 1 & 0 & 0 & \dots & u^N \end{pmatrix} \tag{4}$$

for the N-component case.

In [8], we have completed the generalized gas dynamics hierarchy by presenting the missing equations in the hierarchy together with the Hamiltonian operators as well as the new infinite sets of conserved quantities. We have also started the multi-component systems by presenting the bi-Hamiltonian structure of $N = 3, 4$ cases in (4) and have conjectured that the N-component Kodama equations are bi-Hamiltonian systems.

In this work we shall give a simple method of constructing the second Hamiltonian operator for the N-component Kodama equations (2) and (4). The construction will rely mainly on the dimensional analysis of the equations. This analysis will enable us to write an ansatz for an $N \times N$ matrix γ

$$\gamma = \begin{pmatrix} (N-1)u^1 & (N-2)u^2 & (N-3)u^3 & \dots & u^{N-1} & 0 \\ (N-1)u^2 & (N-2)m^{22} & (N-3)m^{23} & \dots & m^{2,N-2} & 0 \\ (N-1)u^3 & (N-2)m^{23} & (N-3)m^{33} & \dots & m^{3,N-2} & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ (N-1)u^{N-1} & (N-2)m^{2,N-2} & (N-3)m^{3,N-2} & \dots & m^{N-2,N-2} & 0 \\ (N-1)u^N & (N-2)m^{N-1,2} & (N-3)m^{N-1,3} & \dots & m^{N-1,N-2} & N/2 \end{pmatrix} \tag{5}$$

which completely determines the second Hamiltonian operator (see (30)). The matrix γ contains $(N+1)(N-2)/2$ unknown rational functions $m^{ij} = m^{ji}$. We shall show that the monomials in these functions can be determined by solving a linear Diophantine equation for the triple (i, j, N) in the set of positive integers and the construction of the second Hamiltonian operator is thereby reduced to the determination of the coefficients of these monomials from algebraic equations. The number of these constants will be related to the partition functions of linear Diophantine equations. For $N = 2$ the second Hamiltonian operator of shallow-water equations (3) follows from the definition of ansatz. Besides the examples which were already given in [8], we shall work out the $N = 5$ case as a further illustration.

2. Kodama equations: first Hamiltonian structure

Equation (2) combined with (4) can be written as

$$u_t^k = \delta^{k,N} u_x^k + \sum_{i=0}^{N-k} u^{N-i} u_x^{k+i} \qquad k = 1, 2, \ldots, N \tag{6}$$

which are indeed in the form of conservation laws

$$u_t^k = \partial_i Q^k u_x^i \qquad \partial_i = \frac{\partial}{\partial u^i} \tag{7}$$

with appropriate fluxes $Q^k(u)$ for the kth equation, so that each component of the N-vector $\{u^i\}$ is a conserved quantity. Moreover, we have the quadratic conserved quantity

$$H_1^1 = \tfrac{1}{2} g_{(o)ij} u^i u^j \tag{8}$$

where

$$g_{(o)ij} = g_{(o)}^{ij} = (g_{(o)ij})^{-1} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ . & . & \ldots & . & . \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{9}$$

with $\det(g_{(o)ij}) = -1$, is the metric tensor associated with the first Hamiltonian structure which we shall define below. The function H_1^1 will serve as the first Hamiltonian function for the second Hamiltonian structure.

These elementary conserved quantities are evident if we consider the zeroth-order conservation laws

$$\frac{\mathrm{d}K(u)}{\mathrm{d}t} = \frac{\mathrm{d}G(u)}{\mathrm{d}x}. \tag{10}$$

The elimination of the flux in (10) gives necessary conditions for the conserved densities

$$\partial_k \partial_i K\, v_j^i = \partial_j \partial_i K\, v_k^i \tag{11}$$

which are trivially satisfied by u^is and which follow from (9) for H_1^1.

The first Hamiltonian structure of N-component Kodama equations is an immediate consequence of their conservative form and is defined by the constant-coefficient Hamiltonian operator

$$J_o^{ij} = g_{(o)}^{ij} \frac{\mathrm{d}}{\mathrm{d}x} \tag{12}$$

with $g_{(o)}^{ij}$ given by (9). The corresponding Hamiltonian function H_2^1 can be obtained by solving the system of first-order partial differential equations

$$\partial_k H_2^1 = g_{(o)ki} Q^i. \tag{13}$$

The integrability conditions

$$\partial_k \partial_j H_2^1 = g_{(o)ik}\, \partial_j Q^i = g_{(o)ik}\, v_j^i \tag{14}$$

of (13) are satisfied and hence the first Hamiltonian structure is guaranteed.

3. Dimensional analysis

For the N-component Kodama equations we assign to the variable u^i the dimension

$$[u^i] = \frac{2N-1-i}{N-1} \qquad N > 1 \tag{15}$$

so the highest dimension is two and the lowest is one. We find from the dimensional equations corresponding to (6) that $[x]-[t]=1$. Then in the Hamiltonian equations of motion

$$u^i_t = J^{ij}_o\, \partial_j H^1_2 = J^{ij}_1\, \partial_j H^1_1 \tag{16}$$

the first-order hydrodynamic-type Hamiltonian operator J_1, or more precisely (cf [3]) $\gamma + \gamma^t$ in (30) below, is required to have dimensions

$$[J^{ij}_1] = \frac{2N-i-j}{N-1} \tag{17}$$

which are symmetric in i, j. We see immediately that for the upper-left triangle and cross-diagonal of J_1 we have

$$1 \leqslant [J^{ij}_1] \leqslant 2 \qquad \text{for} \quad i+j \leqslant N+1 \tag{18}$$

which indicates a linear dependence on u^is, while for the lower-right triangle

$$0 \leqslant [J^{ij}_1] < 1 \qquad \text{for} \quad i+j > N+1 \tag{19}$$

which is excluded by the range of $[u^i]$s. In order to satisfy (19), we must either have an operator with zeros in the lower-right triangle, except J^{NN}_1 which has constant coefficient, or introduce variables with dimension less than one. Obviously, the first case is a very restrictive class of Hamiltonian operators even without reference to any particular hydrodynamical system. Moreover, it is easy to prove that such an operator does not satisfy the Jacobi identities.

Thus we are led to define the ratios of polynomials in u^i. To this end we note that the variables with a maximal number of different dimensions in between zero and one can be obtained by defining the $N-2$ ratios

$$\xi^i = \frac{u^i}{u^{N-1}} \qquad 1 \leqslant i \leqslant N-2 \tag{20}$$

with dimensions satisfying

$$0 < \frac{1}{N-1} \leqslant [\xi^i] = \frac{N-1-i}{N-1} \leqslant \frac{N-2}{N-1} < 1\,. \tag{21}$$

In fact, with these definitions we have a complete set of dimensions

$$[\text{constant}], [\xi^{N-2}], [\xi^{N-3}], \ldots, [\xi^1], [u^N], \ldots, [u^1] \tag{22}$$

ranging from zero to two in steps of $1/(N-1)$. Note the absence of the quotient u^N/u^{N-1} which has negative dimension.

So far we have clarified the dimensional considerations for why the second Hamiltonian operator, not only for $N = 3,4$, which we have given in [8], but also for generic N, involves the inverse powers of u^{N-1}. These are manifestations of the restrictions imposed by the Jacobi identities.

Now we shall describe the construction of the ansatz for J_1 with dimensions (17) and involving the polynomials in the variables u^i, ξ^i whose dimensions form a complete set. These rational polynomials will contain the monomials of the form

$$(u^1)^{a_1}(u^2)^{a_2}\dots(u^N)^{a_N}(\xi^1)^{\alpha_1}(\xi^2)^{\alpha_2}\dots(\xi^{N-2})^{\alpha_{N-2}} \qquad 0\leqslant a_i,\alpha_i : \text{integers} \tag{23}$$

in a particular entry J_1^{ij} for given N. The monomials (23) are required to satisfy the dimensional relations

$$2N-i-j = 2(N-2)a_1+\cdots+(N-1)a_N+(N-2)\alpha_1+\cdots+2\alpha_{N-3}+\alpha_{N-2}\,. \tag{24}$$

The number of different monomials in J_1^{ij} is the number of different sets of solutions $\{a_1,\dots,a_N,\alpha_1,\dots,\alpha_{N-2}\}$ of (24) for given i,j and N. This is a problem of additive number theory concerning the representations of a given positive integer in the set of positive integers, and (24) is a linear Diophantine equation in this set. The required number of different solutions of (24) is called the partition function $p(2N-i-j)$, and is generated by the product

$$\prod_{k=1}^{2N-2}\frac{1}{(1-t^k)} = \sum_{m=1}^{\infty} p(m)\,t^m \tag{25}$$

which is a formula due to Euler. The partition function depends on the sum $i+j$ and hence the entries of J_1 along the lines drawn parallel to the cross-diagonal involve the same monomials with unknown coefficients. The number of entries of J_1 having the same ansatz is given by

$$\begin{aligned} &i+j-1 && \text{if} \quad i+j\leqslant N+1\\ &2N-(i+j-1) && \text{if} \quad i+j > N+1 \end{aligned} \tag{26}$$

corresponding to the upper-left triangle including cross-diagonal and the lower-right triangle, respectively. Thus, the total number of constants in a general ansatz for J_1 will be

$$\sum_{i+j=2}^{N+1}(i+j-1)p(2N-i-j)+\sum_{i+j=N+2}^{2N}(2N-i-j+1)p(2N-i-j)\,. \tag{27}$$

However, the conditions on J_1 to define a Hamiltonian structure compatible with the first one have immediate consequences which we shall discuss in the next section.

4. Bi-Hamiltonian structure

The analysis in the last section is sufficient to write an ansatz for a matrix of differential operators having as many constants as given by (27). This operator is going to define the bi-Hamiltonian structure of the N-component Kodama equations provided it gives (2) via (16), and it satisfies the Jacobi identities. We shall also require the second Hamiltonian operator to be compatible with (12). In particular, we can infer from the compatibility conditions that γ^{1i} and γ^{i1} must depend linearly on u^i only. Surprisingly, this is also the restriction on the second Hamiltonian operator if we require the Hamiltonian structures for the N-component case to reduce to those of shallow-water equations [9] in the $N = 2$ limit. This limit is achieved by the transformations

$$(u^1, u^2, \ldots, u^{N-1}, u^N) \longmapsto (u^1, 0, \ldots, 0, u^N) \tag{28}$$

which produce singular Jacobians $\mathcal{J}$ for the transformations of Hamiltonian operators according to the rule

$$J \longmapsto \mathcal{J} J \mathcal{J}^{\mathrm{t}} . \tag{29}$$

Evidently, we have found that the second Hamiltonian operators of Kodama equations arise from a framework where the integers a_i are all set to zero except one when $i + j \leqslant N + 1$ is satisfied. In this latter case all α_is are zero. This reduces the number of unknown constants greatly. We shall further give the N^2 of the constants in the ansatz as coefficients of the linear terms in u^i and ξ^i. To this end we write the Hamiltonian operator J_1 in a suitable form

$$J_1 = (\gamma + \gamma^{\mathrm{t}}) \frac{\mathrm{d}}{\mathrm{d}x} + \gamma_x \tag{30}$$

using the $N \times N$ matrix γ given, in the final form, by (5) with superscript t denoting the transpose. This form is an immediate consequence of the skew-adjointness on arbitrary matrix of first-order differential operators. In the differential-geometric terminology of Dubrovin and Novikov (30) is the Liouville–Poisson-type hydrodynamic operator defined on the N-dimensional phase spaces when the coordinates u^i are conserved quantities [3].

Constructing the functions in γ according to the procedure of the last section and determining the unknown constants from Jacobi identities we thus obtain the second Hamiltonian structure. Then by the celebrated theorem of Magri [10] the bi-Hamiltonian structure enables us to generate infinite sequences of conserved quantities. For N-component Kodama equations repeated application of the recursion operator

$$R^i_k = J_1^{ij} \, g_{(o)jk} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^{-1} \tag{31}$$

give $(N - 1)$ sets of infinities of these functions starting from the elementary conserved densities u^i, $i = 1, 2, \ldots, N - 1$ with u^N being the trivial Casimir for each N. As we have shown in [8] for $N = 2, 3$, the missing Nth sequence may start/end from a non-trivial Casimir.

From the dimension of the recursion operator we see that the ith set of conserved quantities start with the dimension $[u^i]$ and proceed by increasing the dimension by one at each recursive step. Thus the nth conserved quantity in the ith sequence have the dimension

$$[H_n^i] = \frac{(n+2)N-(n+1)-i}{N-1}\,. \tag{32}$$

Accordingly we can write an ansatz for H_n^i and determine the constant coefficients of monomials from the conservation equation (10) instead of solving the recursion relations. The first sequence $\{H_n^1\}$ of conserved quantities which starts from u^1 consists of polynomials not containing the inverse powers of u^{N-1}. The Hamiltonian functions of the bi-Hamiltonian structure belong to this sequence.

The construction of conserved quantities is further simplified by noting that $\partial/\partial u^N$ acts as the inverse of the recursion operator for each sequence. Thus once the ansatz for the nth function is solved the conserved densities previous to it can be obtained by applying $\partial/\partial u^N$ repeatedly.

5. Example: $N = 5$

The second Hamiltonian operators for 3- and 4-component Kodama equations were obtained in [8]. As a further example we consider the five-component case for which we have three rational variables ξ^i, $i=1,2,3$. To determine the ansatz we look for the solutions of the equations

$$10-i-j = 3\alpha_1+2\alpha_2+\alpha_3 \tag{33}$$

for $i+j=2,3,\ldots,10$, the last of which is trivial. Each different solution set $\{\alpha_1,\alpha_2,\alpha_3\}$ will give a monomial in J_1^{ij} when used as powers of ξ^i. The number of these monomials will be determined from the Euler formula (25)

$$(1-t)^{-1}(1-t^2)^{-1}(1-t^3)^{-1} = 1+t+2t^2+3t^3+4t^4+5t^5+7t^6+8t^7+\cdots \tag{34}$$

from which we read that there are seven different solutions for $i+j=4$. These are $(0,0,6)$, $(0,1,4)$, $(0,2,2)$, $(0,3,0)$, $(1,0,3)$, $(1,1,1)$, $(2,0,0)$. The corresponding monomials involving rational variables in J_1^{22} are obtained by using these numbers as exponents of ξ^is. Together with the linear dependence on u^3 we get the following ansatz for J_1^{22} containing eight arbitrary constants:

$$J_1^{22} = a_1u^3 + a_2(\xi^3)^6 + a_3\xi^2(\xi^3)^4 + a_4(\xi^2)^2(\xi^3)^2 + a_5(\xi^2)^3 + a_6\xi^1(\xi^3)^3 + a_7\xi^1\xi^2\xi^3 + a_8(\xi^1)^2\,. \tag{35}$$

There is no other entry of J_1 with the same ansatz. Similarly, we obtain five different solutions of (33) for $i+j=5$ and construct the monomials

$$(\xi^3)^5 \qquad \xi^2(\xi^3)^3 \qquad (\xi^2)^2\xi^3 \qquad \xi^1(\xi^3)^2 \qquad \xi^1\xi^2 \tag{36}$$

for an ansatz for J_1^{32} and J_1^{23}. After having the the form of entries with this essentially combinatoric procedure, it remains to determine the constants from the Jacobi identities. In spite of the great simplifications achieved by our knowledge of the entries of the Hamiltonian operator, this is a tedious algebraic computation. We give the final form of second Hamiltonian operator for the five-component Kodama equation

$$J_1 = \frac{1}{8}\begin{pmatrix} 8u^1 & 7u^2 & 6u^3 & 5u^4 & 4u^5 \\ 7u^2 & 6m^{22} & 5m^{23} & 4m^{24} & 3m^{25} \\ 6u^3 & 5m^{23} & 4m^{33} & 3m^{34} & 2m^{35} \\ 5u^4 & 4m^{24} & 3m^{34} & 2m^{44} & m^{45} \\ 4u^5 & 3m^{25} & 2m^{35} & m^{45} & 5 \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{8}\begin{pmatrix} 4u^1_x & 3u^2_x & 2u^3_x & u^4_x & 0 \\ 4u^2_x & 3m^{22}_x & 2m^{23}_x & m^{24}_x & 0 \\ 4u^3_x & 3m^{23}_x & 2m^{33}_x & m^{34}_x & 0 \\ 4u^4_x & 3m^{24}_x & 2m^{34}_x & m^{44}_x & 0 \\ 4u^5_x & 3m^{25}_x & 2m^{35}_x & m^{45}_x & 0 \end{pmatrix} \tag{37}$$

where

$$\begin{aligned} m^{22} &= u^3 - \frac{1}{2}\frac{(u^1)^2}{(u^4)^2} + 2\frac{u^1u^2u^3}{(u^4)^3} - \frac{u^1(u^3)^3}{(u^4)^4} - 3\frac{(u^2)^2(u^3)^2}{(u^4)^4} \\ &\quad + \frac{1}{3}\frac{(u^2)^3}{(u^4)^3} + 3\frac{u^2(u^3)^4}{(u^4)^5} - \frac{5}{6}\frac{(u^3)^6}{(u^4)^6} \\ m^{23} &= u^4 - \frac{u^1u^2}{(u^4)^2} + \frac{u^1(u^3)^2}{(u^4)^3} + 2\frac{(u^2)^2u^3}{(u^4)^3} - 3\frac{u^2(u^3)^3}{(u^4)^4} + \frac{(u^3)^5}{(u^4)^5} \\ m^{33} &= u^5 - \frac{u^1u^3}{(u^4)^2} - \frac{(u^2)^2}{(u^4)^2} + 3\frac{u^2(u^3)^2}{(u^4)^3} - \frac{5}{4}\frac{(u^3)^4}{(u^4)^4} \\ m^{24} &= u^5 - \frac{u^1u^3}{(u^4)^2} - \frac{1}{2}\frac{(u^2)^2}{(u^4)^2} + 2\frac{u^2(u^3)^2}{(u^4)^3} - \frac{3}{4}\frac{(u^3)^4}{(u^4)^4} \\ m^{25} &= \frac{u^1}{u^4} - \frac{u^2u^3}{(u^4)^2} + \frac{1}{3}\frac{(u^3)^3}{(u^4)^3} \qquad m^{34} = \frac{u^1}{u^4} - 2\frac{u^2u^3}{(u^4)^2} + \frac{(u^3)^3}{(u^4)^3}, \\ m^{35} &= \frac{u^2}{u^4} - \frac{1}{2}\frac{(u^3)^2}{(u^4)^2} \qquad m^{44} = \frac{u^2}{u^4} - \frac{(u^3)^2}{(u^4)^2} \qquad m^{45} = \frac{u^3}{u^4}. \end{aligned} \tag{38}$$

6. Conclusions

We are witnessing an interesting hierarchy of Hamiltonian operators starting from $N = 2$ shallow-water structure. We believe that the combinatoric character of this paper will be useful in the investigation of structures of the multicomponent Hamiltonian systems in general. We have kept the construction more general than we needed for Kodama equations since it can be applied to any system where the variables admit scaling properties. The Jacobi identities are thereby reduced to an exercise in linear algebra for the determination of the unknown coefficients. Moreover, the construction of Hamiltonian operators with Hamiltonian functions involving rational polynomials now becomes an easy task.

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